

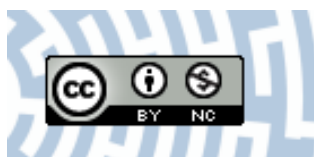


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On affine functions with respect to some means

AGATA NOWAK

Abstract. The purpose of the present paper is to investigate the functional equation

$$M(f(x), g(y)) = h(N(x, y)),$$

where f, g and h are self-mappings of a real interval I and $M, N: I^2 \rightarrow I$ are functions. In particular, we will show that under appropriate assumptions imposed on the functions M, N the local boundedness of f implies the continuity of g .

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1. Introduction: basic definitions and auxiliary lemmas

In this paper I will always denote a non-degenerate interval contained in \mathbb{R} .

A function $M: I^2 \rightarrow I$ is called a *mean* on I if

$$\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}, \quad x, y \in I.$$

A mean M is called a *strict mean* if

$$\min\{x, y\} < M(x, y) < \max\{x, y\}, \quad x, y \in I, x \neq y.$$

Now, let M be a mean on I and let $f: I \rightarrow I$. We say that f is *affine with respect to M* (or shortly *M -affine*) if

$$f(M(x, y)) = M(f(x), f(y)), \quad x, y \in I.$$

Let us also introduce the following notation. For nonempty sets X, Y, Z , $u \in X$, $v \in Y$ and a function $F: X \times Y \rightarrow Z$, we define functions $F_u: Y \rightarrow Z$, $F^v: X \rightarrow Z$ by the formulas:

$$\begin{aligned} F_u(y) &:= F(u, y), & y \in Y; \\ F^v(x) &:= F(x, v), & x \in X. \end{aligned}$$

The present paper is devoted to the following functional equation:

$$M(f(x), g(y)) = h(N(x, y)), \quad x, y \in I, \quad (1.1)$$

where M and N are given functions, $M, N: I^2 \rightarrow I$, whereas f, g and h are unknown functions, $f, g, h: I \rightarrow I$. Our results are analogous to the ones of Ng [1], who has investigated the functional equation

$$f(x) + g(y) = h(T(x, y)),$$

where the unknown functions f, g and h act on a connected or a locally connected topological space X , and $T: X^2 \rightarrow X$ is a given function. The methods we apply to (1.1) are modifications of those used by Ng in [1].

In particular, our results cover the case of L -affine functions, where L is a logarithmic mean, i.e.

$$L(x, y) := \begin{cases} \frac{x-y}{\ln x - \ln y}, & x \neq y, \\ x, & x = y, \end{cases}$$

which was investigated by Matkowski in [2].

Let us start with a technical lemma.

Lemma 1. *Let $M: I^2 \rightarrow I$ be a function such that for all $u, v \in I$ the mapping M_u is strictly increasing, and the mapping M^v is increasing and continuous. Then, for all $s, S, v_0 \in I$ such that $s \leq S$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $u, \bar{u}, v \in I$ satisfying $M(u, v) = M(\bar{u}, v_0)$, $\bar{u} - u \leq \delta$ and $s \leq u \leq S$, we have $v - v_0 \leq \varepsilon$.*

Proof. Suppose (in search of a contradiction) that there exist $s, S, v_0 \in I$ such that $s \leq S$ and

$$\exists \varepsilon > 0 \forall \delta > 0 \exists u, \bar{u}, v \in I \quad (M(u, v) = M(\bar{u}, v_0), \bar{u} - u \leq \delta, s \leq u \leq S, v - v_0 > \varepsilon).$$

In particular, for every $n \in \mathbb{N}$ there exist $u_n, \bar{u}_n, v_n \in I$ fulfilling the conditions:

$$M(u_n, v_n) = M(\bar{u}_n, v_0), \bar{u}_n - u_n \leq \frac{1}{n}, s \leq u_n \leq S, v_n - v_0 > \varepsilon.$$

Observe that the sequence $(u_n)_{n \in \mathbb{N}}$ has to have a convergent subsequence, thus we may assume that the sequence is convergent to some $u \in I$.

If $u_n + \frac{1}{n} \in I$ for infinitely many $n \in \mathbb{N}$, then we deduce the following inequalities:

$$M(u_n, v_0 + \varepsilon) \leq M(u_n, v_n) = M(\bar{u}_n, v_0) \leq M\left(u_n + \frac{1}{n}, v_0\right).$$

On letting n tend to infinity, we get that $M(u, v_0 + \varepsilon) \leq M(u, v_0)$, which contradicts the fact that M_u is strictly increasing.

If $u_n + \frac{1}{n} \in I$ holds only for finitely many $n \in \mathbb{N}$, then $u = S \in I$ is the right endpoint of the interval I . Then we have:

$$M(u_n, v_0 + \varepsilon) \leq M(u_n, v_n) = M(\bar{u}_n, v_0) \leq M(u, v_0)$$

and letting n tend to infinity, we get that $M(u, v_0 + \varepsilon) \leq M(u, v_0)$, which contradicts again the fact that M_u is strictly increasing. \square

The proof of the next lemma is analogous and therefore we omit it here.

Lemma 2. *Let $M: I^2 \rightarrow I$ be a function such that for all $u, v \in I$ the mapping M_u is strictly increasing, and the mapping M^v is increasing and continuous. Then, for all $s, S, v_0 \in I$ such that $s \leq S$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $u, \bar{u}, v \in I$ satisfying $M(u, v) = M(\bar{u}, v_0)$, $u - \bar{u} \leq \delta$ and $s \leq \bar{u} \leq S$, we have $v_0 - v \leq \varepsilon$.*

In the proof of our main result we also need the following lemma:

Lemma 3. [3] *Let X be a connected and locally connected space, $\theta: X \rightarrow \mathbb{R}$ a continuous function and $t_1, t_2 \in \theta(X)$, $t_1 < t_2$. Then there exists a connected subset $B \subseteq \theta^{-1}((t_1, t_2))$ such that $\theta(B) = (t_1, t_2)$.*

2. Main results

Now let us state and prove our main result:

Theorem 1. *Let $M, N: I^2 \rightarrow I$ be functions such that for all $u, v \in I$ the mapping M_u is strictly increasing, the mapping M^v is strictly increasing and continuous, and the mappings N_u, N^v are continuous. Assume that a triple $(f, g, h): I \rightarrow I^3$ is a solution of (1.1) and there exists a subinterval $I_0 \subseteq I$ such that f is nonconstant on I_0 and $f(I_0) \subseteq [s, S]$ for some $s, S \in I$. Then g is continuous.*

Proof. Choose $x_1, x_2 \in I_0$ such that $f(x_1) \neq f(x_2)$. From (1.1) and the strict monotonicity of $M^{g(y)}$ it follows that $N(x_1, y) \neq N(x_2, y)$ for all $y \in I$.

Fix $y_0 \in I$ arbitrarily. We will prove the continuity of g at y_0 using Lemmas 1 and 2 for $v_0 = g(y_0)$. Let $\varepsilon > 0$; there exists a $\delta > 0$ such that the conditions of Lemmas 1 and 2 hold. Denote $l_1 := N(x_1, y_0)$ and $l_2 := N(x_2, y_0)$. We may assume that $l_1 < l_2$. According to Lemma 3 applied to $X = I_0$, $\theta = N^{y_0}$, $t_1 = l_1$, $t_2 = l_2$ there exists an interval $B \subseteq I_0$ such that $N(B, y_0) = (l_1, l_2)$. The function f is bounded on I_0 and $B \subseteq I_0$, so $\sup\{f(x) : x \in B\} < +\infty$. Hence, $f(x_0) \geq f(x) - \delta$ for some $x_0 \in B$ and for all $x \in B$.

Let $V := \{y \in I : l_1 < N(x_0, y) < l_2\}$. It is easy to notice that $y_0 \in V$ and V is open, since N_{x_0} is continuous. For arbitrary $y \in V$ we have $N(x_0, y) \in (l_1, l_2) = N(B, y_0)$, so there exists an $x \in B$ such that $N(x_0, y) = N(x, y_0)$. By virtue of (1.1) we get $M(f(x_0), g(y)) = M(f(x), g(y_0))$. Apply Lemma 1 to $v_0 := g(y_0)$, $s, S, \bar{u} := f(x)$, $u := f(x_0)$, $v := g(y)$ to get that $g(y) - g(y_0) \leq \varepsilon$ for $y \in V$.

Since $N(x_0, y_0) \in (l_1, l_2) = N(B, y_0)$ we can find points $x_3, x_4 \in B$ fulfilling $N(x_3, y_0) < N(x_0, y_0) < N(x_4, y_0)$. Define the set W by

$$W := \{y \in I : N(x_3, y) < N(x_0, y_0) < N(x_4, y)\}.$$

One can check that W is a neighborhood of y_0 . Moreover, for each $y \in W$ the set $N^y([\min\{x_3, x_4\}, \max\{x_3, x_4\}])$ is an interval which contains the points $N(x_3, y)$ and $N(x_4, y)$. Therefore, the point $N(x_0, y_0)$, which lies between them, belongs to that set. Thus, for every $y \in W$ there exists a point x between x_3 and x_4 , which fulfills $N(x_0, y_0) = N(x, y)$. But such a point x must belong to B , since B is connected and $x_3, x_4 \in B$. From (1.1) we get $M(f(x_0), g(y_0)) = M(f(x), g(y))$. From Lemma 2 it follows that $g(y_0) - g(y) < \varepsilon$ for $y \in W$.

Thus, g is continuous at y_0 . Since y_0 is arbitrarily chosen, g is continuous. \square

Remark 1. Let us note that if I is compact, then the assumption of the boundedness of f by elements from I is obviously fulfilled. Similarly, if f is continuous in some point x_0 and $f(x_0) \in \text{int}I$, then f is bounded on some neighborhood of x_0 by elements from I .

Proof. If $I = [a, b]$ and $f: I \rightarrow I$, then the assumptions of the previous theorem are fulfilled for $s = a$, $S = b$, $I_0 = I$.

If f is continuous in some point x_0 and $f(x_0) \in \text{int}I$, then there exist $s, S \in \text{int}I$ such that $s < f(x_0) < S$ and a non-degenerate interval $I_0 \subseteq I$ such that $x_0 \in I_0$ and $f(I_0) \subseteq [s, S]$. \square

In the following corollary we will show that under auxiliary assumptions imposed on functions M, N and if $f = g$, then it is enough to assume that f is bounded from one side on some open subset of I by an element from I .

Corollary 1. *Let $M, N: I^2 \rightarrow I$ be functions such that for all $u, v \in I$ the mapping M_u is strictly increasing, the mapping M^v is strictly increasing and continuous and the mappings N_u, N^v are strictly increasing, continuous and $N(x, x) = x$ for every $x \in I$. If a pair $(f, h): I \rightarrow I^2$ is a solution of the equation*

$$h(N(x, y)) = M(f(x), f(y)), \quad x, y \in I \quad (2.1)$$

and there exist a subinterval $I_0 \subseteq I$ and a constant $S \in I$ (or $s \in I$) such that $f(I_0) \subseteq (-\infty, S]$ (or $f(I_0) \subseteq [s, +\infty)$, respectively) and f is non-constant on I_0 and M^S is onto I , then f is continuous.

Proof. We show that the function f fulfilling the assumptions of the corollary also fulfills the assumptions of Theorem 1.

Let f be bounded from above on an interval $I_0 = [a, b] \subseteq I$ by a constant $S \in I$ and let $x_0 := \frac{1}{2}(a + b)$. We will prove that there exists a point $x_1 \in (a, x_0)$ such that for every $x \in (x_1, x_0)$ there exists $y \in (a, b)$ such that $N(x, y) = x_0$. First, we show that there exists a pair $(x_1, y_1) \in (a, x_0) \times (a, b)$

such that $N(x_1, y_1) = x_0$. If there were no such pair, then we would have $N(x, (a, b)) \subseteq (a, x_0)$ for each $x \in (a, x_0)$. We infer that for every $\hat{x} \in (x_0, b)$

$$x_0 = N(x_0, x_0) < N(x_0, \hat{x}) = \lim_{x \rightarrow x_0^-} N(x, \hat{x}) \leq x_0,$$

which is impossible. Let $(x_1, y_1) \in (a, x_0) \times (a, b)$ be a pair for which $N(x_1, y_1) = x_0$. For every $x \in (x_1, x_0)$ we get

$$N(x, x_0) < N(x_0, x_0) = x_0, \quad N(x, y_1) > N(x_1, y_1) = x_0.$$

It means that the interval $N(x, (a, b))$ contains a point greater than x_0 and a point smaller than x_0 . Thus, there exists $y \in (a, b)$ such that $N(x, y) = x_0$.

Now let $x \in (x_1, x_0)$. We may find a $y \in (a, b)$ for which $N(x, y) = x_0$. Use Eq. (2.1) for the pair (x, y) to get

$$h(x_0) = h(N(x, y)) = M(f(x), f(y)) \leq M(f(x), S).$$

Since $h(x_0) \in I$ and M^S is a bijection from I onto I , there exists exactly one $s \in I$ such that $M^S(s) = h(x_0)$. Observe that $s \leq f(x)$ for every $x \in (x_1, x_0)$. Thus, f and $g = f$ satisfy the assumptions of Theorem 1.

The proof when $f(I_0) \subseteq (s, \infty)$ follows in a similar fashion. \square

On imposing stronger assumptions upon N we may eliminate the assumption of non-constancy of f on every interval.

Theorem 2. *Let $M, N: I^2 \rightarrow I$ be mappings such that for all $u, v \in I$ the mapping M_u is strictly increasing, the mapping M^v is strictly increasing and continuous and the functions N_u, N^v are continuous. Moreover, let N be a symmetric strict mean. Assume that a triple $(f, g, h): I \rightarrow I^3$ is a solution of (1.1) and that there exist a subinterval $I_0 \subset I$ and $s, S \in I$ such that $f(I_0) \subset [s, S]$. Then g is continuous.*

Proof. Due to Theorem 1 it is enough to consider the case when f is constant on some non-degenerate subinterval $P \subseteq I$. We prove that in this case g is constant on the whole domain.

Let $c := f|_P$; we have

$$\begin{aligned} M(c, g(x)) &= M(f(y), g(x)) = h(N(y, x)) = h(N(x, y)) \\ &= M(f(x), g(y)) = M(c, g(y)), \quad x, y \in P. \end{aligned}$$

Since the function M_c is strictly monotone, g is constant on P , say $g|_P \equiv c_1$ for some $c_1 \in I$. Thus, for all $x, y \in P$ we get

$$h(N(x, y)) = M(f(x), g(y)) = M(c, c_1),$$

which means that $h|_{N(P \times P)}$ is constant and equal to $c_2 := M(c, c_1)$. On the other hand, N is a mean, thus $N(P \times P) = P$. Thus, we have shown that $h|_P \equiv c_2$.

Now we will verify that f is constant on $\text{int}I$. Assume that P is the maximal interval on which f is constant and equals c (if Q is any interval such that $P \subseteq Q$ and $f|_Q \equiv c$ then $P = Q$); we will show that $\text{int}P = \text{int}I$. Let $\alpha, \beta \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ be chosen in such a way that $\text{int}I = (\alpha, \beta)$. Suppose that $\text{int}P \neq \text{int}I$. Then the left end-point of P is a real number $a > \alpha$ or the right end-point of P is a real number $b < \beta$. Without loss of generality we may assume that $b < \beta$. Choose d so that $[d, b) \subseteq P$. We have $(b, b + \frac{1}{n}) \subseteq I$ for sufficiently large $n \in \mathbb{N}$. For such n we may choose $y_n \in (b, b + \frac{1}{n})$ such that $f(y_n) \neq c$. The mean N is strict, so $d < N(d, b) < b$. Moreover, we have $N(d, u) \in (d, b)$ for all points u taken from some neighbourhood of b because the function N_d is continuous. Thus, $N(d, y_k) \in [d, b)$ for some $k \in \mathbb{N}$. On the other hand,

$$M(f(y_k), c_1) = M(f(y_k), g(d)) = h(N(y_k, d)) = c_2.$$

Thus, $f(y_k) = c$, because $M(c, c_1) = c_2$ and the function M^{c_1} is strictly increasing. Thus, we get a contradiction.

Obviously, $g|_{\text{int}I} \equiv c_1$ and $h|_{\text{int}I} \equiv c_2$.

To complete the proof it remains to show that if any of the end-points of I belongs to I , then in this edge g is also equal to c_1 . For example, if $\beta \in I$, then from Eq. (2.1) applied to $x \in (\alpha, \beta)$ and $y = \beta$ we get

$$c_2 = h(N(x, \beta)) = M(f(x), g(\beta)) = M(c, g(\beta)),$$

thus, $g(\beta) = c_1$. If $\alpha \in I$, we proceed in a similar way. □

In what follows we apply the just proved theorems to M -affine functions. From results of Ng [1] it follows that an M -affine function defined on I and bounded from both sides has to be continuous. From the previous results it follows that two-sided boundedness of f may be weakened to one-side boundedness.

Corollary 2. *If M is a symmetric, strict mean such that the functions M_u, M^v are continuous and strictly increasing for every $u, v \in I$ and f is an M -affine function, then the local boundedness of f from one side by an element from I implies its continuity.*

In particular, setting the logarithmic mean for M we obtain a slightly more general result than those of Matkowski [2, Lemma 2].

Corollary 3. *Assume that I is contained in $(0, +\infty)$ and L is the logarithmic mean. If $f: I \rightarrow I$ is L -affine, bounded at a point from one side by an element from I then f is continuous.*

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